2019-Nov-12 (pre-lecture) MAT244 Prof. Yun William Yu

Monday, November 11, 2019 11:41 AM

Outline

- Matrix ODEs
- Classification of 2D linear systems with constant coefficients
- Inhomogeneous Systems

Last time:

Given $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, $A \in M_{n\times n}(\mathbb{R})$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,

$$x(t) = \exp(tA)\, x_0.$$

This time:

Suppose we can find a matrix P s.t. $A = P \Lambda P^{-1}$, for Λ diagonal and the columns of P eigenvectors of A.

Then $x(t) = \exp(tA)x_0 = \exp(tP\Lambda P^{-1})x_0 = P\exp(t\Lambda)P^{-1}x_0$.

Let $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, so $v_1, \dots, v_n$ are eigenvectors,

and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues

Then $P\exp(t\Lambda) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \cdot \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda_1 t} & \cdots & v_n e^{\lambda_n t} \end{bmatrix}$

And let $P^{-1}x_0 = y_0 = \begin{bmatrix} y_{0,1} \\ \vdots \\ y_{0,n} \end{bmatrix}$

Then $x(t) = \begin{bmatrix} v_1 e^{\lambda_1 t} & \cdots & v_n e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} y_{0,1} \\ \vdots \\ y_{0,n} \end{bmatrix} = y_{0,1} v_1 e^{\lambda_1 t} + \cdots + y_{0,n} v_n e^{\lambda_n t}$

Thus, all solutions are linear combinations of the eigenvectors of A times $e^{\lambda_i t}$.

Ex. $\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We want to find the eigenvalues & eigenvectors.

Trick from linear algebra

$$\left\{\begin{array}{l} \text{Given a } 2\times 2 \text{ matrix } A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \qquad Tr(A)=a+d, \qquad \det(A)=ad-bc. \\ \text{Then } \quad \lambda^2 - Tr(A)\cdot\lambda + \det(A)=0. \end{array}\right.$$

$$\left(\begin{vmatrix} \lambda-a & b \\ c & \lambda-d \end{vmatrix} = \lambda^2-(a+d)\lambda+ad-bc = \lambda^2 - TrA + \det(A) \right)$$

$$\Rightarrow \quad \lambda^2-\lambda-2=0$$

$$(\lambda-2)(\lambda+1)=0$$

$$\lambda=2,-1.$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow \left.\begin{array}{l} x_1+2x_2=2x_1 \\ x_1 = 2x_2 \end{array}\right\} v_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_1=2$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \Rightarrow \left.\begin{array}{l} x_1+2x_2=-x_1 \\ x_1 = -x_2 \end{array}\right\} v_2=\begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \lambda_2=-1$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Method 1:

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{-t} \\ e^{2t} + e^{-t} \end{bmatrix}$$

Method 2: Ansatz: $\quad x(t) = c_1\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t}$

$$x(0) = \begin{bmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_1=1,\ c_2=1$$

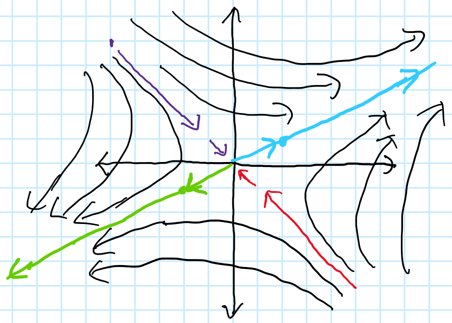
$$\Rightarrow x(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-t} = \begin{bmatrix} 2e^{2t} - e^{-t} \\ e^{2t} + e^{-t} \end{bmatrix}$$

Phase portrait:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \dot{x} = Ax.$$

Then $\lambda_1 = 2$, $\lambda_2 = -1$

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{So} \quad x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$



Similar to slope field.

Suppose $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

$x(0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \Rightarrow x(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x(t) = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$x(0) = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \Rightarrow x(t) = 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

The qualitative behavior of the ODE given different starting points is captured by the phase portrait and completely determined by the eigenvalues + eigenvectors.

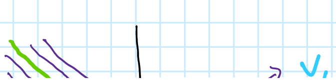
Cases:

2 real eigenvalues
- ↳ both positive
- ↳ both negative
- ↳ mix of positive + negative

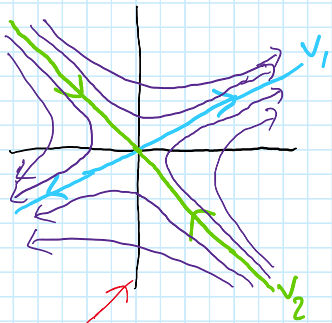
1 real eigenvalue of multiplicity 2
- ↳ 2 linearly ind. eigenvectors
- ↳ 1 linearly ind. eigenvector

2 complex conjugate eigenvalues
- ↳ real part negative
- ↳ real part = 0
- ↳ real part positive

Case: 2 real eigenvalues



$\lambda_1 > 0 \qquad x(t) = c_1 \cdot v_1 \cdot e^{\lambda_1 t} + c_2 v_2 \cdot e^{\lambda_2 t}$

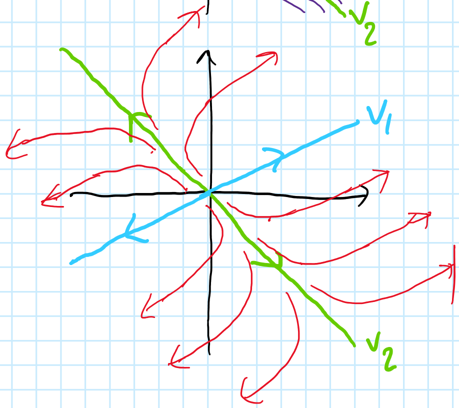


$\lambda_1 > 0$

$\lambda_2 < 0$

$x(t) = c_1 \cdot v_1 \cdot e^{\lambda_1 t} + c_2 v_2 \cdot e^{\lambda_2 t}$

Saddle point

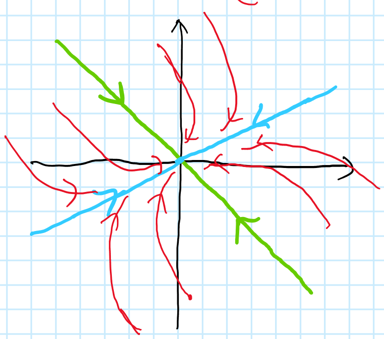


$\lambda_1 > 0$

$\lambda_2 > 0$

$|\lambda_1| > |\lambda_2|$

Source. Everything goes away from the origin, but the component along v_1 gets magnified more quickly



$\lambda_1 < 0$

$\lambda_2 < 0$

$|\lambda_2| > |\lambda_1|$

Sink. Everything goes to the origin, but the component along v_2 decays more quickly

Case: 1 real eigenvalue

Suppose we have 2 linearly ind. eigenvectors v_1, v_2.

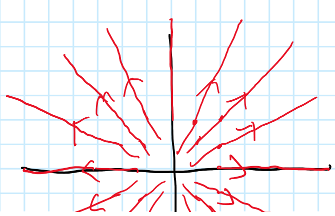
Then $A = P \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} P^{-1}$, where $P = [v_1 \;\; v_2]$.

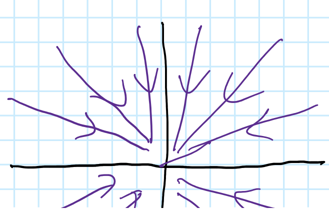
But diagonal matrices commute with all other matrices

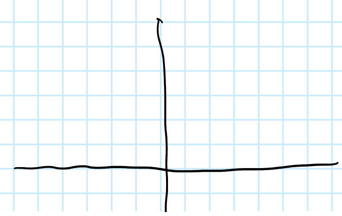
So $A = P P^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

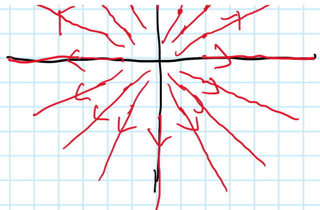
Then for $\dot{x} = Ax$, we get $x = \exp(\lambda A) x_0 = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} x_0 = e^{\lambda t} x_0$

So $x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda t} = e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$



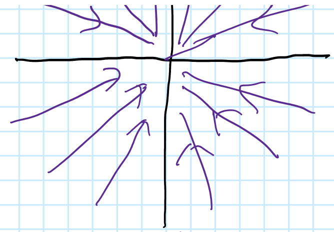






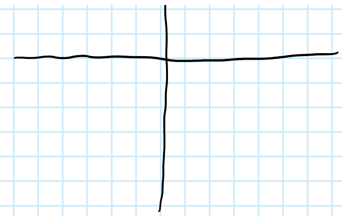
If $\lambda > 0$

source



If $\lambda < 0$

sink



If $\lambda = 0$.

What if we only have 1 eigenvector?

e.g. $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ eigenvalue $\lambda = 3$

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} \quad \left\{ \begin{matrix} 3x_1 + x_2 = 3x_1 \\ x_2 = 3x_2 \end{matrix} \right\} \quad x_2 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\dot{x} = Ax \qquad x(0) = x_0$

$\Rightarrow \quad x = \exp(tA)x_0$

$$\exp(tA) = \exp\left(t\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right)$$

Note $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, so the two matrices commute.

Thus, $\exp(tA) = \exp\left(t\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) \cdot \exp\left(t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$$= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \cdot \exp\left(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right).$$

Recall $\exp(X) = I + X + \dfrac{X^2}{2!} + \dfrac{X^3}{3!} + \dfrac{X^4}{4!} + \cdots$

Note $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

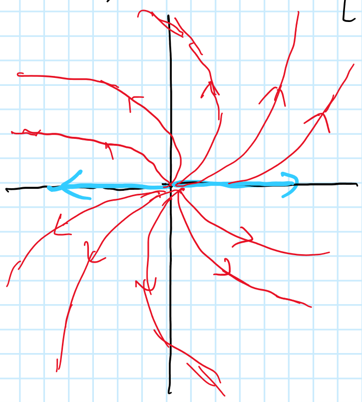
So $\exp\left(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \cancel{\dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{2!}} + \cancel{\dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{3!}} + \cdots$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

$\Rightarrow \exp(tA) = \begin{bmatrix} e^{3t} & 0 \\ \cdot & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & \cdot \end{bmatrix} = \begin{bmatrix} e^{3t} & te^{3t} \\ \cdot & \cdot \end{bmatrix}$

$$\Rightarrow \exp(tA) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

Thus, $x(t) = c_1 \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} te^{3t} \\ e^{3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} te^{3t} \\ e^{3t} \end{bmatrix}$



Try using slope field

$$\dot{x} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 3 \end{bmatrix} x_2$$

If we start on the eigenvector, we stay on the eigenvector. Otherwise, the solution has weird behavior due to the extra polynomial t term.

Complicated, and I won't ask you to draw this case.

Case: Two complex conjugate eigenvalues

Then we don't have real eigenvectors, but we do have complex ones.

Let $\lambda = \alpha \pm \beta i$ be the two complex eigenvalues of A, $(\lambda_1^* = \lambda_2)$
and let v_1, v_2 be the corresponding complex eigenvectors.

Because v_1, v_2 are eigenvectors to conjugate eigenvalues, they are also conjugate WLOG.
i.e. $v_1^* = v_2$.

Let $P = [v_1 \;\; v_2]$.

Then $A = P \Lambda P^{-1}$, where $\Lambda = \begin{bmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{bmatrix}$.

Let $\dot{x} = Ax$

Then $x = \exp(tA)\, x_0$, where $x_0 = x(0)$.

$x(t) = P \exp(t\Lambda) P^{-1} x_0$. Say $y_0 = P^{-1} x_0$, the coordinates in this other basis

Then $x(t) = [v_1 \;\; v_2] \begin{bmatrix} e^{(\alpha+\beta i)t} & 0 \\ 0 & e^{(\alpha-\beta i)t} \end{bmatrix} \begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix}$

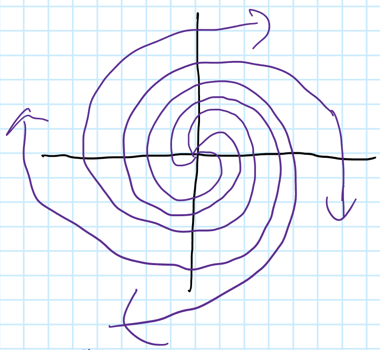
$$x(t) = v_1 e^{(\alpha+\beta i)t} \cdot y_{0,1} + v_2 e^{(\alpha-\beta i)t} y_{0,2}$$

In order for the final solution to be real for all t, $y_{0,1}^* = y_{0,2}$.

Thus $x(t) = 2\,Re\left(v_1 e^{(\alpha+\beta i)t} y_{0,1}\right)$

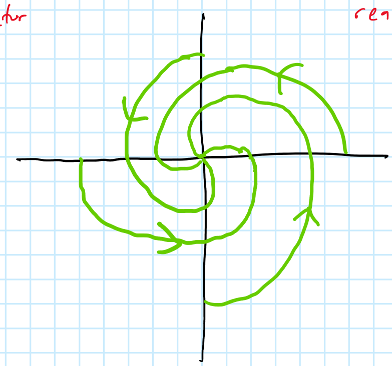
$$= 2\,Re\left(v_1 y_{0,1} e^{\alpha t} \cos(\beta t) + v_1 y_{0,1} e^{\alpha t} i \sin(\beta t)\right)$$

$$= 2\cos(\beta t)\, e^{\alpha t} \cdot \underbrace{Re(v_1 y_{0,1})}_{\text{real vector}} - 2\sin(\beta t)\, e^{\alpha t} \underbrace{Im(v_1 y_{0,1})}_{\text{real vector}}$$



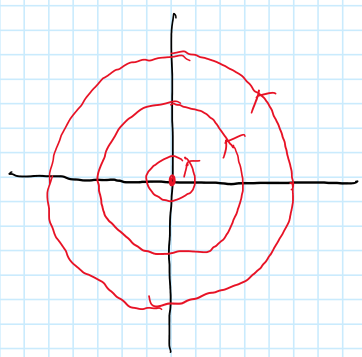
If $\alpha > 0$

spiral source



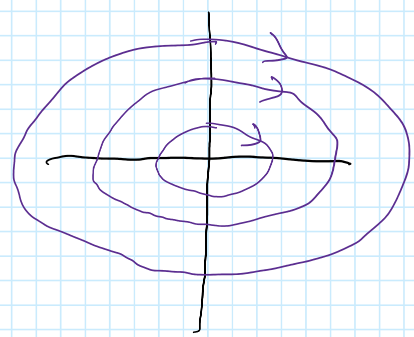
If $\alpha < 0$

spiral sink



If $\alpha = 0$

center



Terminology: A linear system is stable if all solutions are bounded, and asymptotically stable if all solutions converge to 0 as $t \to \infty$.

Inhomogeneous systems:

Given $\dot{x}(t) = Ax(t) + g(t)$, $x(0) = x_0$,

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)\, g(s)\, ds. \qquad \text{(Teschl 3.48)}$$

Recall given $x^{(n)} + c_{n-1}x^{(n-1)} + \dots + c_1\dot{x} + c_0 x = g(t)$, HW, Tesch 3.57

$$x(t) = x_h(t) + \int_0^t u(t-s)\, g(s)\, ds$$

So you can explicitly solve the inhomogeneous eqn. using this formula.

So you can explicitly solve the inhomogeneous eqn. using this formula.

But remember, there are other ways to solve linear ODEs with constant coefficients, such as undetermined coefficients, variation of parameter, series methods, etc. These all still work here.

Ex.

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

From above, $\lambda_1 = 2 \quad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So

$$x_c = c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Guess $x_p = \begin{bmatrix} k_1 t + k_2 \\ k_3 t + k_4 \end{bmatrix}$ (method of undetermined coefficients)

Then

$$\dot{x}_p = \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 t + k_2 \\ k_3 t + k_4 \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \begin{bmatrix} k_1 t + k_2 + 2k_3 t + 2k_4 + 1 \\ k_1 t + k_2 + t \end{bmatrix}$$

$$\begin{cases} k_1 = t(k_1 + 2k_3) + (k_2 + 2k_4 + 1) \\ k_3 = t(k_1 + 1) + k_2 \end{cases}$$

$$\begin{cases} k_1 + 2k_3 = 0 \\ k_2 + 2k_4 + 1 = k_1 \\ k_2 = k_3 \\ k_1 + 1 = 0 \end{cases} \Rightarrow \begin{array}{l} k_1 = -1 \\ k_2 = \frac{1}{2} \\ k_3 = \frac{1}{2} \\ k_4 = -\frac{5}{4} \end{array} \Rightarrow x_p = \begin{bmatrix} -t + \frac{1}{2} \\ \frac{1}{2} t - \frac{5}{4} \end{bmatrix}$$

$$\Rightarrow x = x_c + x_p = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} -t + \frac{1}{2} \\ \frac{1}{2} t - \frac{5}{4} \end{bmatrix}$$